A q, r-analogue for the Stirling numbers of the second kind of Coxeter groups of type B

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Stirling numbers of type A and classical identities

The most common combinatorial interpretation of **Stirling number of the second kind** $S(n,k) = \left\{ n \atop k \right\}$ is as counting the number of set partitions of $[n] = \{1, \dots, n\}$ into k blocks.

Theorem (classical)

The Stirling numbers of the second kind satisfy the following recursion:

$$\left\{ n \atop k \right\} = \left\{ n-1 \atop k-1 \right\} + k \left\{ n-1 \atop k \right\},$$

with the boundary condition: $\left\{ n \atop n \right\} = 1$, $\left\{ n \atop 0 \right\} = 0$.

Stirling numbers of type A and classical identities

Theorem (classical)

Let $x \in \mathbb{R}$ and let $n \in \mathbb{N}$. Then:

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} [x]_k$$

where $[x]_k = x(x-1)\cdots(x-k+1)$ is the **falling polynomial of degree k** *and $[x]_0 = 1$.*

Definition

Definition: A **set partition of $[n]$ of type B** is a way to divide the set $\{\pm 1, \ldots, \pm n\}$ into blocks such that:

- If B appears as a block in the partition, then $-B$ (which is obtained by negating all the elements of B) also appears in that partition.
- There exists at most one block satisfying $-B = B$. This block is called the **zero block** (if it exists, it is a subset of $\{\pm 1, \ldots, \pm n\}$ of the form $\{\pm i \mid i \in C\}$ for some $C \subseteq [n]$).

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Example

$$\{\underbrace{\{2,-2,4,-4\}}_{\text{zero block}}, \underbrace{\{1,-3,6,8,-9\}}_{B_1}, \underbrace{\{-1,3,-6,-8,9\}}_{-B_1}, \underbrace{\{-5,7\}}_{B_2}, \underbrace{\{5,-7\}}_{-B_2}\}.$$

A bit of history

Stirling numbers also count the number of elements of a constant rank in the intersection lattice of hyperplane arrangements of Coxeter type A.

Carlitz (1933): Definition of q-Stirling for type A.

Dowling (1973), Zaslavsky (1981): Implicit introduction of set partitions of type B (in the form of signed graphs).

Dolgachev-Lunts, Reiner (90'): Explicit generalization of set partitions to hyperplane arrangements of Coxeter groups of type B.

Broder (1984): A variant of Stirling numbers, which counts the set partitions such that the first r elements are placed in r distinct parts of the set partition.

Sagan-Swanson (2022): A q-Stirling numbers for type B (of both kinds).

Definition (Hutchinson (1963), Milne (1977))

The word $\omega_1 \cdots \omega_n$ over the alphabet $[\pm n] = \{1, \ldots, n\}$ is called a **restricted growth (RG)-word** if $\omega_1 = 1$ and for each $2 \leq t \leq n$ one has: $\omega_t \leq \max\{\omega_1, \ldots, \omega_{t-1}\} + 1$.

Example

$\omega = 122123$ is an RG-word, but $\omega = 14213$ is not.

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Let $P = \{B_1, \ldots, B_k\}$ be a set partition of $[n]$ whose blocks are ordered in such a way that the set of minimum elements of the blocks is increasing. We associate an RG-word $\omega_1 \cdots \omega_n$ such that ω_j is the number of the block where j is located.

Example

The RG-word $\omega = 122123$ is matched with $\{\{1, 4\}, \{2, 3, 5\}, \{6\}\}$.

Restricted growth words of type B

Definition

Let $\Sigma^B = \{0, \pm 1, \pm 2, \ldots, \pm n\}$ and define an order on Σ^B:

$$0 \prec -1 \prec 1 \prec -2 \prec 2 \prec \cdots \prec -n \prec n.$$

A **restricted growth (RG-)word of type B** is a word $\omega = \omega_1 \cdots \omega_n$ over Σ^B which satisfies the following conditions:

(1) We have $\omega_1 = 0$ or $\omega_1 = 1$.

(2) For each $2 \leq t \leq n$, the following inequality holds:

$$\omega_t \preceq \max\{\omega_1, \ldots, \omega_{t-1}\} + 1.$$

In the case that

$$\omega_t = \max\{\omega_1, \ldots, \omega_{t-1}\} + 1,$$

we demand: $\omega_t > 0$.

Denote by $R^B(n, k)$ the set of RG-words of type B over Σ^B such that the positive maximal element is k.

Let $P = \{B_0, B_1, \ldots, B_k\}$ be a set partition of $[\pm n]$ of type B, such that its representative blocks are ordered in such a way that the set of minimum positive elements of the blocks is increasing (where B_0 is the zero-block, if exists). Associate an RG-word $\omega = \omega_1 \cdots \omega_n$ of type B as follows:

- ω_j is the number of the representative block where j or $-j$ is located.
- If the element j appears in the representative block, then ω_j is the number of the block containing j.
- Otherwise, ω_j is the number of this block, with a negative sign.

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Example

Given: $P = \{B_0 = \{2, 5, -2, -5\},\ B_1 = \{1, -7\},\ B_2 = \{3, -4, 6\}\}$, its associated RG-word of type B is:

$$\omega = (1, 0, 2, -2, 0, 2, -1)$$

q-Stirling number of second kind of type B

Definition (Motivated by Cai-Readdy (2017) for type A)

Let $\omega = \omega_1 \cdots \omega_n \in R^B(n,k)$. Define the **weight of ω** by $\mathrm{wt}(\omega) = \prod_{i=1}^{n} \mathrm{wt}_i(\omega)$, where $\mathrm{wt}_1(\omega) = 1$ and for $2 \le i \le n$,

$$\mathrm{wt}_i(\omega) = \begin{cases} 1 & |\omega_i| > \max\{|\omega_1|, \ldots, |\omega_{i-1}|\} \text{ or } \omega_i = 0 \\ q^{2|\omega_i|-1} & |\omega_i| \le \max\{|\omega_1|, \ldots, |\omega_{i-1}|\} \text{ and } \omega_i < 0 \\ q^{2|\omega_i|} & |\omega_i| \le \max\{|\omega_1|, \ldots, |\omega_{i-1}|\} \text{ and } \omega_i > 0 \end{cases}$$

$\left\{ n \atop k \right\}_{[q]}^B := \sum_{\omega \in R^B(n,k)} \mathrm{wt}(\omega)$ - **q-Stirling number of second kind of type B**.

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$\left\{ {n \atop k} \right\}^B_{[q]} := \sum_{\omega \in R^B(n,k)} \mathrm{wt}(\omega)$ - **q-Stirling number of second kind of type B**.

Example

Given: $\{B_0 = \{2,-2\}, B_1 = \{1,-3\}, B_2 = \{4,-5,6\}\}$,

its associated RG-word is: $\omega = (1,0,-1,2,-2,2)$, so we have:

$$\mathrm{wt}(\omega) = \underbrace{1}_{\mathrm{wt}_1} \cdot \underbrace{1}_{\mathrm{wt}_2} \cdot \underbrace{q}_{\mathrm{wt}_3} \cdot \underbrace{1}_{\mathrm{wt}_4} \cdot \underbrace{q^3}_{\mathrm{wt}_5} \cdot \underbrace{q^4}_{\mathrm{wt}_6} = q^8$$

The recursion for q-Stirling number

Remark

$$\left\{ {n \atop k} \right\}^B_{[q=1]} = \left\{ {n \atop k} \right\}^B.$$

Theorem (BGK, 2022)

Let $[k]_q := \frac{1-q^k}{1-q} = 1 + q + \cdots + q^{k-1}$. *For each* $1 \leq k < n$,

$$\left\{ {n \atop k} \right\}^B_{[q]} = \left\{ {n-1 \atop k-1} \right\}^B_{[q]} + [2k+1]_q \cdot \left\{ {n-1 \atop k} \right\}^B_{[q]},$$

with the boundary conditions: $\left\{ {n \atop 0} \right\}^B_{[q]} = \left\{ {n \atop n} \right\}^B_{[q]} = 1.$

The r-variant of the q-Stirling numbers

Definition

Let $R_r^B(n,k)$ be the subset of $R^B(n,k)$ consisting of all RG-words such that the first r entries are $1,2,\ldots,r$ in increasing order.

$\left\{ n \atop k \right\}_{[q],r}^B := \sum_{\omega \in R_r^B(n,k)} \mathrm{wt}(\omega)$ - is the **q,r-Stirling number $\left\{ n \atop k \right\}_{[q],r}^B$ of the second kind of type B**.

This subset corresponds to set partitions of type B where the first r elements are in r distinct blocks.

The recursion for the r-variant of the q-Stirling numbers

Theorem (BGK, 2022)

For each $n > r$ and $1 \leq k < n$,

$$\left\{ {n \atop k} \right\}^B_{[q],r} = \left\{ {n-1 \atop k-1} \right\}^B_{[q],r} + [2k+1]_q \cdot \left\{ {n-1 \atop k} \right\}^B_{[q],r},$$

with the boundary conditions:

$$\text{if } n < r, \text{ then } \left\{ {n \atop k} \right\}^B_{[q],r} = 0;$$

$$\text{if } n = r, \text{ then } \left\{ {n \atop k} \right\}^B_{[q],r} = \delta_{k,r};$$

$$\text{if } n > r, \text{ then } \left\{ {n \atop 0} \right\}^B_{[q],r} = \left\{ {n \atop n} \right\}^B_{[q],r} = 1.$$

The ordinary generating functions of $\left\{ {n \atop k} \right\}_{[q]}^{B}$ and $\left\{ {n \atop k} \right\}_{[q],r}^{B}$

Theorem (BGK, 2022)

For all $k \in \mathbb{N}$, the ordinary generating function of $\left\{ {n \atop k} \right\}_{[q]}^{B}$ is:

$$\sum_{n=0}^{\infty} \left\{ {n \atop k} \right\}_{[q]}^{B} x^n = \frac{x^k}{(1-x)(1-[3]_q x)\cdots(1-[2k+1]_q x)}.$$

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The exponential generating function of $\left\{ {n+r \atop k+r} \right\}_r^B$ and its application to bases of $\mathbb{R}[x]$

Theorem (BGK, 2022)

Let $r, k \in \mathbb{N}$. *Then:*

$$\sum_{n=k}^{\infty} \left\{ {n+r \atop k+r} \right\}_r^B \frac{x^n}{n!} = \frac{1}{k!2^k} e^{(2r+1)x} (e^{2x} - 1)^k.$$

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Theorem (BGK, 2022)

Let $n, r \in \mathbb{N}$. Then we have:

$$\begin{aligned}(x+2r)^n &= \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r^B \cdot 2^k \left(\frac{x-1}{2}\right)^{\underline{k}} \\ &= \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r^B \cdot (x-1)(x-3)\cdots(x-2k+1).\end{aligned}$$

Thank you for your attention!!