Block numbers, 321-avoidance and Schur-positivity

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Short description of results

We present here three results concerning the block number statistic on 321-avoiding permutations:

- Equi-distribution of block number and the complement of last descent over certain sets of 321-avoiding permutations.
- The set of 321-avoiding permutations with a given block number is symmetric and Schur-positive.
- An explicit formula for the corresponding character.

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- An explicit formula for the corresponding character.

Outline

Introduction

Pattern avoiding

- Let S_n be the symmetry group on $[n] = \{1, 2, \ldots, n\}$.
- For $\pi \in S_n$ and $\tau \in S_k$, we say that π contains τ if π has a sub sequence which is order-isomorphic to τ.
- Otherwise, π avoids τ.

Example

41523 avoids the pattern 321 but contains the pattern 132 since its sub sequence 153 is order-isomorphic to 132 but no sub sequence of 41523 is order-isomorphic to 321.

Describing pattern-avoiding classes

Let $S_n(\Pi)$ be the set of permutations in S_n avoiding a given set of patterns Π. There are several ways to provide information about this set.

1. Compute the cardinality $|\mathcal{S}_n(\Pi)|$ (Simion, Wilf, ...).
2. Compute the generating function for a statistic *stat*:

$$\sum_{\pi \in \mathcal{S}_n(\Pi)} q^{stat(\pi)}$$

(Sagan, Pak, Elizalde,...).

3. Compute the quasisymmetric function

$$\sum_{\pi \in \mathcal{S}_n(\Pi)} F_\pi(x_1, x_2, \ldots)$$

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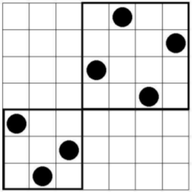
Equi-distribution

Direct sum of permutations

Definition

Let $\pi \in \mathcal{S}_m$ and $\sigma \in \mathcal{S}_n$. The direct sum of π and σ is the permutation $\pi \oplus \sigma \in \mathcal{S}_{m+n}$ defined by

$$(\pi \oplus \sigma)_i = \begin{cases} \pi(i), & \text{if } i \le n; \\ \sigma(i-n)+n, & \text{otherwise.} \end{cases}$$



The permutation $312 \oplus 2413 = 3125746$

Block number

Definition

A nonempty permutation which is not a direct sum of two nonempty permutations is called $\oplus$-irreducible.
Each permutation π can be written uniquely as a direct sum of $\oplus$-irreducible ones, called the blocks of π. Their number $\mathrm{bl}(\pi)$ is the block number of π.

Example

$$\mathrm{bl}(45321) = 1,$$

$$\mathrm{bl}(312 \mid 54) = 2,$$

$$\mathrm{bl}(1 \mid 2 \mid 3 \mid 4) = 4.$$

Remarks

- Direct sums and block decomposition of permutations appear naturally in the study of pattern-avoiding classes (Albert, Atkinson, Vatter).
- The block number of an arbitrary permutation was previously studied by Richard Stanley (2005), as the cardinality of the connectivity set (defined by Comtet).

Last descent

Definition

For a permutation $\pi \in \mathcal{S}_n$ let

$$\mathrm{ldes}(\pi) := \max\{i : \ i \in \mathrm{Des}(\pi)\},$$

with $\mathrm{ldes}(\pi) := 0$ *if* $\mathrm{Des}(\pi) = \emptyset$ *(i.e., if π is the identity permutation).*

Example

$$\mathrm{ldes}(3176245) = 4$$

The sets $Bl_{n,k}$ and $L_{n,k}$

Definition

Let

$$Bl_{n,k} := \{\pi \in \mathcal{S}_n(321) : \ \mathrm{bl}(\pi) = k\}.$$

Note that $\mathrm{bl}(\pi) = \mathrm{bl}(\pi^{-1})$.

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Cardinality

Definition

Recall: The n-th Catalan number is

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The corresponding generating function is

$$c(x) = \sum_{n=0}^{\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}.$$

Cardinality

Definition

For each $k \geq 0$, the n-th k-fold Catalan number is the coefficient of x^n in $(xc(x))^k$. Explicitly:

$$C_{n,k} = \frac{k}{2n-k}\binom{2n-k}{n}.$$

Proposition

For positive integers $n \geq k \geq 1$:

$$C_{n,k} = |SYT(n-1, n-k)| = |L_{n,n-k}| = |B_{n,k}|$$

This result will be refined in the sequel.

Left-to-right maxima

Definition

The set of left-to-right maxima of $\pi \in \mathcal{S}_n$ *is*

$$\mathrm{ltrMax}(\pi) = \{i \mid \pi(i) > \pi(j) \text{ for all } i < j\}$$

Example

$$\pi = \bar{3}12\bar{5}4\bar{6}.$$

Observation

For 321-avoiding permutations, the set of left-to-right maxima determines the descent set. Explicitly, for any $1 \le i \le n-1$,

$$i \in \mathrm{Des}(\pi) \iff i \in \mathrm{ltrMax}(\pi) \text{ and } i+1 \notin \mathrm{ltrMax}(\pi).$$

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Main result 1: Equi-distribution

Theorem (Adin-B.-Roichman '16)

For every positive integer n,

$$\sum_{\pi \in \mathcal{S}_n(321)} \mathbf{x}^{\mathrm{ltrMax}(\pi^{-1})} q^{\mathrm{bl}(\pi)} = \sum_{\pi \in \mathcal{S}_n(321)} \mathbf{x}^{\mathrm{ltrMax}(\pi^{-1})} q^{n-\mathrm{ldes}(\pi)}.$$

Note the analogy with the classical

Theorem (Foata-Schützenberger '70)

For every positive integer n,

$$\sum_{\pi \in \mathcal{S}_n} \mathbf{x}^{\mathrm{Des}(\pi^{-1})} q^{\mathrm{inv}(\pi)} = \sum_{\pi \in \mathcal{S}_n} \mathbf{x}^{\mathrm{Des}(\pi^{-1})} q^{\mathrm{maj}(\pi)}.$$

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Symmetry and Schur-positivity

Symmetric functions

Definition

A symmetric function is a formal power series $f \in \mathbb{C}[[x_1, x_2, \cdots]]$ *which is invariant under any permutation of the variables.*

- We sometimes restrict to a finite number of variables by setting almost all of them to zero.

Example

$f = x_1 + x_2 + x_3$ is symmetric and homogeneous of degree 1. (with $x_4 = x_5 = \ldots = 0$).

Semistandard tableaux

Definition

Let λ be a partition. A semistandard Young tableau of shape λ is a filling of the cells of λ by positive integers such that

- The entries in each row are weakly increasing.
- The entries in each column are strictly increasing.

Example

$$\lambda = (4, 3, 2) \qquad T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 3 \\ \hline 3 & 3 & 4 & \\ \cline{1-3} 5 & 6 & & \\ \cline{1-2} \end{array}$$

Schur functions

With each semistandard Young tableau T we associate a monomial

$$\mathbf{x}^T = \prod_i x_i^{\text{number of i's in } T}.$$

Example

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 3 \\ \hline 3 & 3 & 4 \\ \cline{1-3} 5 & 6 \\ \cline{1-2} \end{array} \qquad \mathbf{x}^T = x_1 x_2^2 x_3^3 x_4 x_5 x_6.$$

The Schur function s_λ associated with a partition λ is defined by

$$s_\lambda = \sum_{T \in SSYT(\lambda)} \mathbf{x}^T.$$

Schur functions

Example

For $\lambda = (2,1)$, the semistandard tableaux of shape λ filled with numbers out of $\{1,2,3\}$ are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\,.$$

The corresponding Schur polynomial is

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$$

Proposition

$\{s_\lambda \mid \lambda \vdash n\}$ is a basis for the vector space of symmetric functions which are homogeneous of degree n.

Schur functions

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Proposition

$\{s_\lambda \mid \lambda \vdash n\}$ is a basis for the vector space of symmetric functions which are homogeneous of degree n.

Schur-positivity

Definition

A symmetric function is called Schur-positive if all the coefficients in its expansion in the basis of Schur functions are non-negative.

Example

For $\lambda \vdash k$ and $\mu \vdash \ell$, consider the product

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu.$$

The Littlewood-Richardson rule provides a combinatorial interpretation of the coefficients $c^\nu_{\lambda,\mu}$, proving that $s_\lambda s_\mu$ is Schur-positive.

An equivalent definition of symmetric functions

A formal power series $f(x_1, x_2, \ldots)$ is symmetric if for every composition $\alpha = (\alpha_1, \ldots, \alpha_n)$, all monomials $x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$ in f with distinct indices have the same coefficient.

Example

$$f = \sum_{i \neq j} x_i^3 x_j = x_1^3 x_2 + x_1 x_2^3 + x_1^3 x_3 + x_1 x_3^3 + x_2^3 x_3 + x_2 x_3^3 + \ldots$$

quasisymmetric functions

A formal power series $f(x_1, x_2, \ldots)$ is quasisymmetric if for every composition $(\alpha_1, \cdots, \alpha_k)$, all monomials $x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$ in f with indices $i_1 < i_2 < \cdots < i_k$ have the same coefficients.

Example

$$f = \sum_{i<j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + \ldots$$

is quasisymmetric but not symmetric.

Denote by $QSym$ the vector space of quasisymmetric functions which are homogeneous of degree n.

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The fundamental basis

For each subset $J \subseteq [n-1]$ define the corresponding fundamental quasisymmetric function

$$F_J(\mathbf{x}) := \sum_{\substack{i_1 \le i_2 \le \ldots \le i_n \\ i_j < i_{j+1} \text{ if } j \in J}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

In particular, J can be the descent set of a permutation.

Example

$\pi = 312$, $\mathrm{Des}(\pi) = \{1\}$.

$$\mathcal{F}_{\mathrm{Des}(312)} = \mathcal{F}_{\{1\}} = x_1x_2x_3 + x_1x_2x_2 + x_1x_3x_3 + x_2x_3x_3.$$

$\pi = 132$, $\mathrm{Des}(\pi) = \{2\}$.

$$\mathcal{F}_{\mathrm{Des}(132)} = \mathcal{F}_{\{2\}} = x_1x_1x_2 + x_1x_1x_3 + x_1x_2x_3 + x_2x_2x_3.$$

Proposition (Gessel)

$\{F_J \mid J \subseteq [n-1]\}$ *is a basis for* $QSym_n$.

s_λ's expansion in fundamental basis

Since s_λ is symmetric, it is also quasysymmetric, so it can be expanded in the F_J basis.

Definition

T is a standard Young tableau.

$$Des(T) = \{i \; i+1 \text{ is in lower row than i }\}$$

Example

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 9 \\ \hline 3 & 4 & 7 \\ \cline{1-3} 6 & 8 \\ \cline{1-2}\end{array} \qquad Des(T) = \{2, 5, 7\}.$$

Theorem

(Gessel) $S_\lambda = \sum_{T \in SYT(\lambda)} F_{Des(T)}$.

Example

$\lambda = (2,1)$.

$$SYT(\lambda) = \{ T_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} \qquad , T_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array} \qquad \}.$$

$Des(T_1) = \{2\}, Des(T_2) = \{1\}$. Hence $s_{(2,1)} = F_1 + F_2$.

$$\mathcal{F}_{\{1\}} = x_1x_2x_3 + x_1x_2x_2 + x_1x_3x_3 + x_2x_3x_3.$$

$$\mathcal{F}_{\{2\}} = x_1x_1x_2 + x_1x_1x_3 + x_1x_2x_3 + x_2x_2x_3.$$

and indeed, recall that

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$$

Schur-positivity

For $A \subseteq \mathcal{S}_n$, let

$$\mathcal{Q}(A) = \sum_{\pi \in A} \mathcal{F}_{Des(\pi)}.$$

$\mathcal{Q}(A)$ is called Schur-positive if it is symmetric and can be written as a linear combination of Schur functions with non-negative coefficients.

Question (Adin-Roichman, '13)

For which $A \subseteq \mathcal{S}_n$ is $\mathcal{Q}(A)$ (symmetric and) Schur-positive?

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Symmetry and Schur-positivity

Classical examples of (symmetric and) Schur-positive sets of permutations include:

- Conjugacy classes
- Inverse descent classes
- Knuth classes
- Permutations with a fixed inversion number
- Arc permutations

Problem (Sagan and Woo, '14)

Find sets of patterns Π *and parameters stat such that* $\mathcal{Q}(\{\sigma \in \mathcal{S}_n(\Pi) \mid stat(\sigma) = k\})$ *is symmetric and Schur-positive.*

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Deeper into Schur positivity of Knuth classes

- The Robinson-Schensted-Knuth (RSK) correspondence maps each permutation $\pi \in \mathcal{S}_n$ to a pair (P_π, Q_π) of standard Young tableaux of the same shape λ.
- A fundamental property of the RSK correspondence is:

Fact

For each $\pi \in \mathcal{S}_n$, $\mathrm{Des}(P_\pi) = \mathrm{Des}(\pi^{-1})$ *and* $\mathrm{Des}(Q_\pi) = \mathrm{Des}(\pi)$.

Knuth classes

Definition

For every standard Young tableau T of size n, the set

$$\mathcal{C}_T := \{\pi \in \mathcal{S}_n : \ P_\pi = T\}$$

is the Knuth class *corresponding to T.*

proposition

Knuth classes are Schur-positive.

Example

$213 \mapsto \left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right)$ and $231 \mapsto \left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right)$, so that

$$\mathcal{C}_{\ldots} = \{213, 231\}$$

Proposition

(Gessel, '84) Knuth classes are Schur-positive.

Inverse descent classes

Definition

For $J \subseteq \mathcal{S}_n$:

$$D_J^{-1} = \{\pi \in \mathcal{S}_n \mid Des(\pi^{-1}) = J\}.$$

Corollary

Inverse descent classes are Schur positive.

Pattern avoidance

The sequences $(a_1, \cdots, a_k)$ and $(b_1, \cdots, b_k)$ are order-isomorphic if for all $i, j \in \{1, \ldots, k\}$

$$a_i < a_j \iff b_i < b_j.$$

Given two permutations $\pi \in \mathcal{S}_n, \sigma \in \mathcal{S}_k$, π contains σ if there is a subsequence of π which is order-isomorphic to σ.

Example

216354 contains 312.

Definition

B is a set of permutations.
π avoids B if it does not contain any $\sigma \in B$.

$$\mathcal{S}_n(B) = \{\pi \in \mathcal{S}_n \mid \pi \text{ avoids } B\}$$

Proposition

For each $\pi \in \mathcal{S}_3$ and $n \in \mathbb{N}$:

$$|\mathcal{S}_n(\pi)| = \frac{1}{n+1}\binom{2n}{n}$$

(Catalan number).

Main result 2: Schur-positivity of $Bl_{n,k}$

Recall

Definition

$$Bl_{n,k} := \{\pi \in \mathcal{S}_n(321) : \ \mathrm{bl}(\pi) = k\}.$$

Theorem (Adin-B.-Roichman '16)

$\mathcal{Q}(Bl_{n,k})$ is (symmetric and) Schur positive.

Main result 2: Schur-positivity of $Bl_{n,k}$

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Main result 3: The character

Recall that the Frobenius image of an $\mathcal{S}_n$-character $\chi = \sum_{\lambda \vdash n} c_\lambda \chi^\lambda$ is the symmetric function $f = \sum_{\lambda \vdash n} c_\lambda s_\lambda$, denoted by $ch(\chi)$.

Theorem (Adin-B.-Roichman '16)

For every positive integer $1 \leq k \leq n-1$

$$\mathcal{Q}(Bl_{n,k}) = ch(\chi^{(n-1,n-k)} \downarrow_{\mathcal{S}_n}^{\mathcal{S}_{2n-k-1}})$$

and, for $k = n$,

$$\mathcal{Q}(Bl_{n,k}) = ch(\chi^{(n)}) = s_{(n)}.$$

Proof idea

Proof idea: bijection

The proofs use an explicit left-to-right-maxima preserving bijection from $Bl_{n,k}$ to $L_{n,n-k}$.

Definition

Define a map $f_n : \mathcal{S}_n(321) \mapsto \mathcal{S}_n(321)$, recursively on n, as follows. Each permutation $\pi \in S_n$ belongs to exactly one of the following 3 classes, distinguished according to the location of the letter n and the relative order of $n-1$ and n.

- L : n is the last letter.
- D: n is not the last letter, and $n-1$ precedes n.
- R: $n-1$ is to the right of n.

Proof idea: bijection

The proofs use an explicit left-to-right-maxima preserving bijection from $Bl_{n,k}$ to $L_{n,n-k}$.

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- L : n is the last letter.
- D: n is not the last letter, and $n-1$ precedes n.
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Proof idea: bijection

Case L: n is the last letter.

- Omit n
- Apply f_{n-1};
- Insert n at the last position.

Case D: $n-1$ is left of n, but n is not the last letter.

- Omit n.
- Apply f_{n-1}.
- Multiply from left by the transposition $(n-k-1, n-k)$.
- Insert n at the same position as in π.

Case R: $n-1$ is right of n.
In this case $n-1$ must be the last letter.

- Exchange $n-1$ and n in π, then omit n.
- Apply f_{n-1}
- Multiply (from the left) the resulting permutation by the cycle $(n-k, n-k+1, ..., n-1, n)$.

Example

Let $\pi_8 = \pi = 31254786$.

$$\pi_8 = 312 \mid 54 \mid 786 \xrightarrow[(45)]{D} \pi_7 = 312 \mid 5476 \xrightarrow[(4567)]{R} \pi_6 = 312 \mid 54 \mid 6$$

$$\xrightarrow{L} \pi_5 = 312 \mid 54 \xrightarrow[(345)]{R} \pi_4 = 312 \mid 4$$

$$\xrightarrow{L} \pi_3 = 312 \xrightarrow[(23)]{R} \pi_2 = 21$$

Example (cont.)

In the other direction:

$$
\begin{aligned}
f(\pi_2) = 21(3) \quad & \xrightarrow{R,(23)} \quad f(\pi_3) = 312 \xrightarrow{L} f(\pi_4) = 3124(5) \\
& \xrightarrow{R,(345)} \quad f(\pi_5) = 41253 \xrightarrow{L} f(\pi_6) = 412536(7) \\
& \xrightarrow{R,(4567)} \quad f(\pi_7) = 5126374 \xrightarrow{D,(45)} f(\pi_8) = 41263785
\end{aligned}
$$