Permutation patterns 20 poster: Block numbers and descents of fully commutative signed permutations

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Block number and last descent

Fully commutative signed permutations

A fully commutative element in a Coxeter group W is $\in W$ such that one can arrive from each reduced decomposition of w to another using only the commuting relations $s_is_j = s_js_i$. for $|i-j|>1$. The set of f.c elements of W is denoted by $FC(W)$.

Example

$FC(S_n) = S_n(321)$.
In B_n, w is fully commutative if and only if w avoids the pattern $(-1,-2)$ and a all patterns (a,b,c) such that $|a|>b>c$.

Block number in S_n

A nonempty permutation of S_n which is not a direct sum of two nonempty permutations is called $\oplus$-irreducible.
Each permutation π can be written uniquely as a direct sum of $\oplus$-irreducible ones, called the blocks of π. Their number (π) is the block number of π.

Block number for signed permutations

For a signed permutation $w \in B_n$, let $\tau(w)$ be the permutation in S_n, which records the letters $w_1,\cdots,w_n$ in the relative standard order.

Example

Let $\pi = [-2,4,3,1,-5]$. Then $\tau(\pi) = [2,4,5,3,1]$.

Last descent and Neg

For $\pi \in B_n$ let $\mathrm{Des}_B(w) := \{0 \le i < n : \ell_B(\pi s_i) < \ell_B(w)\}$ be the Descent set of π where $\{s_0, s_1, \ldots, s_{n-1}\}$ is the Coxeter set of type B_n and ℓ_B is the length function.
For a permutation $\pi \in \mathcal{S}_n$ let

$$\mathrm{ldes}(\pi) := \max\{i : \ i \in \mathrm{Des}(\pi)\},$$

with $\mathrm{ldes}(\pi) := 0$ if $\mathrm{Des}(\pi) = \emptyset$ (i.e., if π is the identity permutation).

Example

$$\mathrm{ldes}(3176245) = 4$$

For $\pi \in B_n$, define $Neg(\pi) = \{i \mid 1 \le i \le n, \pi(i) < 0\}$

Theorem

For any positive integer n we have the following equidistribution on (B_n):

$$\sum_{w\in(B_n)} \mathbf{x}^{\mathrm{Des}_B(w)}\mathbf{z}^{\mathrm{Neg}(w)} q^{(w^{-1})} t^{n-\mathrm{ldes}(w^{-1})} = \sum_{w\in(B_n)} \mathbf{x}^{\mathrm{Des}_B(w)}\mathbf{z}^{\mathrm{Neg}(w)} q^{n-\mathrm{ldes}(w^{-1})} t^{(w^{-1})}.$$

Proof idea

Use the intersection of $FC(B_n)$ with S_n coset decomposition and Robbie's bijection $f: S_n(321) \to S_n(321)$ which sends $bl(w^{-1})$ to $n - ldes(f(w))$, preserving the descent set.

Main result 1: Equi-distribution

Equidistribution

Theorem

For any positive integer n, we have

$$\sum_{w\in FC(B_n)\setminus FC(S_n)} q^{(\tau(w^{-1}))} F^B_{\mathrm{Des}_B(w)} = \sum_{k=1}^{n}\left(\sum_{j=0}^{n} b_{n,k,j} q^j\right) s_{(k)}(x_0,x_1,x_2,\ldots)\, s_{(n-k)}(x_1,x_2,\ldots), \quad (1)$$

where

$$b_{n,k,j} = \#\{T \in \mathrm{SYT}((n,k)/(k)) : \ \mathrm{ldes}(T) = n-j\},$$

which is thus non-negative.

Proof idea:

The proof involves several ingredients:

(i) The description of (B_n) as a disjoint union of two-sided Kazhdan–Lusztig cells, given by Green and Losonczy [?].
(ii) A recent descent set preserving bijection from domino tableaux to bi-tableaux, which expands upon Barbash–Vogan's bijection [?] and [?, ?].
(iii) A type B extension of Rubey's involution [?] on 321-avoiding permutations. The latter is in particular descent set preserving and maps the block number to the last descent position of the inverse.
(iv) A detailed study of the intersection of (B_n) with S_n-cosets. This is done by using the descriptions of fully commutative elements in types A_{n-1} and B_n in terms of heaps, which were given for instance in [?, ?].

Theorem

We have the following decomposition

$$(B_n) = \biguplus_{\pi\in(S_n)} B_n(\pi)\cdot\pi, \quad (2)$$

where

$$B_n(\pi) := \begin{cases} \{\mu \in B_n \mid \boldsymbol{\mu} = \mu_1\cdots\mu_{v(\pi)},\ \mu_i \in \{e,\delta_i\}\} & \text{if } 1 \notin \mathrm{Des}(\pi^{-1}); \\ \{\mu \in B_n \mid \boldsymbol{\mu} \in \{e,\delta_1,\ldots,\delta_{v(\pi)}\}\} & \text{if } 1 \in \mathrm{Des}(\pi^{-1}), \end{cases}$$

*and $v(\pi)$ is the first valley of π from Definition **??**.*

Bi-shapes

A *bi-shape* $(\lambda^+,\lambda^-) \vdash n$ is a pair of partitions of total size n.
The *standard descent set* of a standard Young bi-tableau T of size n is defined as

$$\mathrm{Des}(T) := \{0 < i < n : \ i+1 \text{ is in a lower row than } i\}.$$

The *type B descent set* of a standard Young bi-tableau T of size n is defined as

$$\mathrm{Des}_B(T) := \begin{cases} \mathrm{Des}(T) \sqcup \{0\}, & 1 \in T_\mu, \\ \mathrm{Des}(T), & 1 \in T_\lambda. \end{cases}$$

Example

Here are two standard bi-tableau of shape $((2,1),(2))$:

$$T = \begin{array}{cccc} & & 1 & 3 \\ & & 2 & \\ 4 & 5 & & \end{array}, \qquad P = \begin{array}{cccc} & & 2 & 5 \\ & & 3 & \\ 1 & 4 & & \end{array}$$

with descent sets $\mathrm{Des}(T) = \mathrm{Des}_B(T) = \{1,3\}$ and $\mathrm{Des}(P) = \{2,3\} \subsetneq \mathrm{Des}_B(P) = \{0,2,3\}$.

A *standard domino tableau* of shape $\lambda \vdash 2n$ is a filling of the cells of the diagram of shape λ by the letters $1,\ldots,n$ such that each letter $1 \le i \le n$ fills exactly two adjacent cells, and for each $1 \le k \le n$, the union of the cells filled by the letters $\le k$ forms a Young diagram of ordinary shape. Denote the set of standard domino tableaux of shape λ by (λ).
The *standard descent set* of a standard domino tableau consists of all letters $1 \le i < n$, such that the northeast cell filled by $i+1$ is in a lower row than the northeast cell filled by i. Denote the letter in the (i,j) cell of by $_{i,j}$. The *type B descent set* of a standard domino tableau of size n is defined as

Symmetry and Schur-positivity

Quasi-symmetric functions

A formal power series $f(x_1,x_2,\ldots)$ is quasi-symmetric if for every composition $(\alpha_1,\cdots,\alpha_k)$, all monomials $x_{i_1}^{\alpha_1}\cdots x_{i_k}^{\alpha_k}$ in f with indices $i_1 < i_2 < \cdots < i_k$ have the same coefficients.

Example

(In 3 variables)

$$f = x_1^2x_2 + x_1^2x_3 + x_2^2x_3$$

is quasi-symmetric but not symmetric.

Example

$$f = \sum_{i<j} x_i^2x_j = x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + \ldots$$

is quasi-symmetric but not symmetric.
Denote by $QSym$ the vector space of quasi-symmetric functions which are homogeneous of degree n.

The Chow's fundamental B-type basis

For an infinite set of formal variables $x_0,x_1,x_2,\ldots,$ *the type B Chow fundamental quasi-symmetric function* indexed by $J \subseteq \{0\} \cup [n-1]$ is defined as

$$F^B_J := \sum_{\substack{0\le i_1\le i_2\le\cdots\le i_n \\ j\in J \Rightarrow i_j<i_{j+1}}} x_{i_1}\cdots x_{i_n}.$$

Theorem

For every partition $\lambda \vdash 2n$,

$$\sum_{\in(\lambda)} F^B_{\mathrm{Des}_B()} = s_{\lambda^+}(x_1,x_2,\ldots)\, s_{\lambda^-}(x_0,x_1,\ldots).$$

Gessel ['84] proved:

$$\{F_J \mid J \subseteq [n-1]\} \text{ is a basis for } QSym_n.$$

Schur-positivity

For $A \subseteq \mathcal{S}_n$, let

$$\mathcal{Q}(A) = \sum_{\pi\in A} \mathcal{F}_{Des(\pi)}.$$

$\mathcal{Q}(A)$ is called Schur-positive if it is symmetric and can be written as a linear combination of Schur functions with non-negative coefficients.

Main result 2: Schur-positivity of $Bl_{n,k}$

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