Continuity of the major index on involutions

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Background

Definition

- S_n is the symmetric group on n elements.
- An involution is an element $\tau \in S_n$ such that $\tau^2 = \epsilon$.
- I_n is the set of all involutions in S_n.

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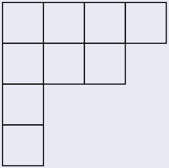
Definition

- A Young diagram is a finite collection of cells in the plane, arranged in left-justified rows, such that the row lengths are increasing.
- The sequence listing the numbers of cells in each row gives a partition λ of a non-negative integer n. The Young diagram is said to be of shape λ.
- λ' is the transpose of λ.

Examples

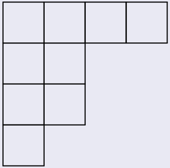
Example

$\lambda =$



$$\lambda = (4, 3, 1, 1)$$

$\lambda' =$



$$\lambda' = (4, 2, 2, 1)$$

Young Tableaux

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- A tableau is **standard** if the numbers are increasing through rows and columns.

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Definition

For a shape λ, let $SYT(\lambda)$ be the set of standard Young tableaux of shape λ.

Example

1	4	5	6
2	7	9	
3			
8			

Descent and major index for permutations

Definition

For $\pi \in S_n$

$$Des(\pi) = \{i \in [n-1] \mid \pi(i) > \pi(i+1)\}$$

$$maj(\pi) = \sum_{i \in Des(\pi)} i$$

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Example

$$Des(3176245) = \{1, 3, 4\}$$

$$maj(3176245) = 1 + 3 + 4 = 8$$

Descent and major index for tableaux

Definition

The *descent set* of T is

$$(T) := \{i \mid i+1 \text{ appears in a lower row of } T \text{ than } i\}.$$

Definition

Define also the *major index* of a standard Young tableau T by

$$maj(T) = \sum_{i \in Des(T)} i.$$

Example

1	4	5	**6**
2	**7**	9	
3			
8			

A standard tableau of shape

$T \in SYT(\lambda = (4,3,1,1)), Des(T) = \{1,2,6,7\}, maj(T) = 1+2+6+7 = 16$

The RSK correspondence

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Fact

For every permutation $\pi \in S_n$,

$$Des(P_\pi) = Des(\pi^{-1}) \quad \text{and} \quad Des(Q_\pi) = Des(\pi).$$

Restriction to involutions

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- Note that π is an involution if and only if $P_\pi = Q_\pi$.
- By restricting the RSK to I_n we get a $Des-$ preserving bijection from I_n to the set of standard Young tableaux of order n, $SYT(n)$.

Example

Let $\pi = 2143 \in I_4$. Then

$$P_\pi = Q_\pi = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

$$Des(\pi) = \{1, 3\} = Des(Q_\pi)$$

Conjugacy classes of involutions

Simple Facts

1. Conjugacy classes in S_n are determined by their cycle structures, which are partitions of n.

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2. The conjugacy classes of involutions in S_n are of the form $(2^k, 1^r)$ such that $0 \leq r \leq n$, $0 \leq k \leq \frac{n}{2}$ and $2k + r = n$.

Theorem (Schützenberger 77')

An involution $\pi \in I_n$ has r fixed points if and only if P_π has r columns of odd length.

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2. The conjugacy classes of involutions in S_n are of the form $(2^k, 1^r)$ such that $0 \leq r \leq n$, $0 \leq k \leq \frac{n}{2}$ and $2k + r = n$.
3. In other words, conjugacy classes of involutions are determined by the number of fixed points.

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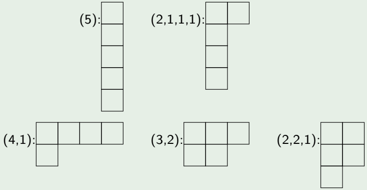
The set $D_n(r)$

Definition

- The set of Young diagrams of size n having exactly r odd columns will be denoted by $D_n(r)$.
- The set of standard Young tableaux of shapes taken from $D_n(r)$ is denoted $SYT_n(r)$.

Example ($D_5(1)$)



(1)

In summary...

Theorem

Let C_μ be the conjugacy class of the partition $\mu = (2^k, 1^r)$. Then the restriction of the RSK correspondence

$$R : C_\mu \to SYT_n(r)$$

is a bijection which preserves the major index, i.e. for each $\pi \in C_\mu$ we have $maj(\pi) = maj(R(\pi))$.

Auxiliary numbers

Definition

For a shape $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_u)$, let

$$b(\lambda) = \sum_{i=0}^{u} i\lambda_i.$$

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Example

$\lambda =$

0	1	2	3
0	1	2	
0	1		

, $\lambda' =$

0	0	0	0
1	1	1	
2	2		

Continuity and extreme values inside each diagram

Theorem (Billey, Kovalinka, Swanson)

Let λ be a Young diagram. Then we have:

$$m(\lambda) := Min\{maj(T) \mid T \in SYT(\lambda)\} = b(\lambda).$$

$$M(\lambda) := Max\{maj(T) \mid T \in SYT(\lambda)\} = \binom{n}{2} - b(\lambda').$$

Moreover, every value between $m(\lambda)$ and $M(\lambda)$ appears at least once except in the case when λ is a rectangle with at least two rows and columns, in which case the values $m(\lambda)+1$ and $M(\lambda)-1$ are missing.

Extreme values over $D_n(r)$

Theorem (B,K 21')

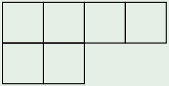
Let $n = 2k + r$.

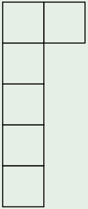
1. *The minimum value of the major index on $D_n(r)$ is k. It is attained by the diagram $\lambda = (n-k, k)$.*
2. *The maximum value of the major index on $D_n(r)$ is $\binom{n}{2} - \binom{r}{2}$. It is attained by the* **odd hook** *$\lambda = (r, 1^{2k}) = (n-2k, 1^{2k})$.*

Example

Maximal value, Odd hook

Minimal value





Minimal gap inside each diagram

Theorem (B,K 21')

1. *Let* $n \geq 6$ *and let* $\lambda \vdash n$ *such that* $\lambda \neq (1^n)$ *and* $\lambda \neq (n)$. *Then*

$$M(\lambda) - m(\lambda) \geq 4.$$

2. *if* $\lambda = (a^b)$ *is a rectangle then*

$$M(\lambda) - m(\lambda) \geq 6.$$

Our main result

Theorem

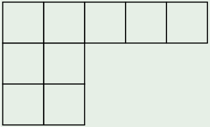
Let $\mu = (2^k, 1^r)$ be a partition of n and let C_μ be the corresponding conjugacy class of involutions in S_n. Then

- *If $r \neq 0$ then the major index on C_μ attains all values between k and $\binom{n}{2} - \binom{r}{2}$.*
- *If $r = 0$ then it attains all the values above excluding $k+1$ and $\binom{n}{2} - 1$.*

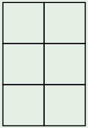
Moreover, any other value outside this range is not attained.

Example

Conj. class with 3 fixed points:



Conj. class without fixed points:



The algorithm

- Traverse $D_n(r)$, starting with $\lambda^0 = (n-k, k)$, which attains the minimum value, k, of *maj* over $D_n(r)$ which is k.

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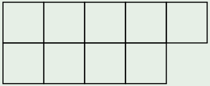

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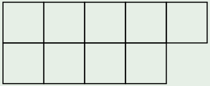
$\lambda^0 =$ 

- End with the odd hook diagram $\lambda^e = (n-2k, 1^{2k})$, attaining the maximum value of *maj* over $D_n(r)$ which is $\binom{n}{2} - \binom{r}{2}$.

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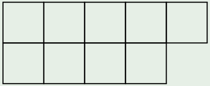
$\lambda^0 =$ 

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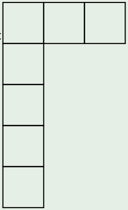
$\lambda^e =$ 

The algorithm Cont.

- The algorithm traverses the set $D_n(r)$ in such a way that in each step one or two squares of a diagram $\lambda \in D_n(r)$ are transferred to a new place to obtain a diagram $\nu \in D_n(r)$ such that

$$M(\lambda) \geq m(\nu)$$

where $M(\lambda)$ ($m(\nu)$) is the maximum (minimum) value of *maj* on $SYT(\lambda)$ ($SYT(\nu)$), respectively

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  where $M(\lambda)$ ($m(\nu)$) is the maximum (minimum) value of *maj* on $SYT(\lambda)$ ($SYT(\nu)$), respectively
- We exclude the hooks from the discussion here. Odd hooks are our final cases while even hooks are treated separately.

Case I: Double stair

$\lambda = (\lambda_1, \ldots, \lambda_u, 0, \ldots, 0)$ contains a consecutive sequence of rows of strictly descending length:

$$\lambda_i > \lambda_{i+1} > \lambda_{i+2} \geq 0.$$

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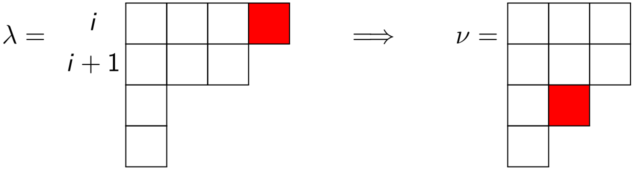
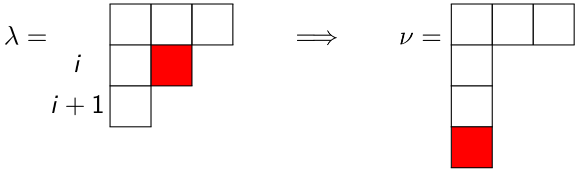
- Take the maximal such sequence.
- This means that λ has columns of lengths i and $i+1$.
- Move the last square of row i to the end of row $i+2$ (which might be empty) to form ν.
- The number of odd columns had not been changed. (columns of lengths i,$i+1$ changed to columns of lengths $i-1, i+2$).



Another example



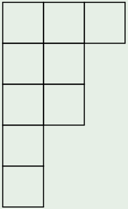
Case II: Remove a domino

A consecutive sequence of rows of strictly descending length does not exist.

- In this case there must exist some i such that $\lambda_i > \lambda_{i+1} = \lambda_{i+2}$

Example

$\lambda =$



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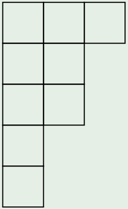

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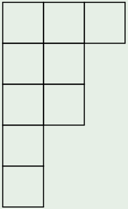

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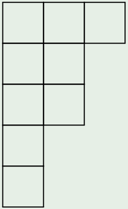
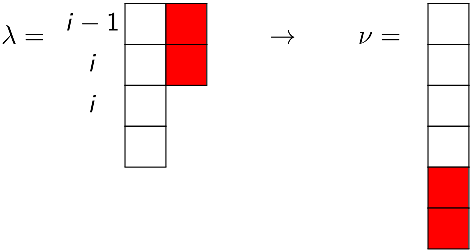


- Choose i to be maximal with respect to this property.
- We must have $\lambda_{i-1} = \lambda_i$, otherwise this case was already treated earlier.
- This means that the squares at the end of rows $i-1$ and i form a vertical domino.

1. If $\lambda_i - \lambda_{i+1} = 1$ then by the maximality of i we must have $\lambda_i = 2$. Transfer the domino to the end of the first column.



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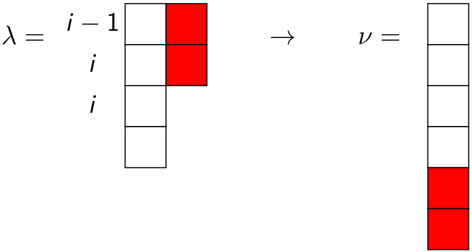
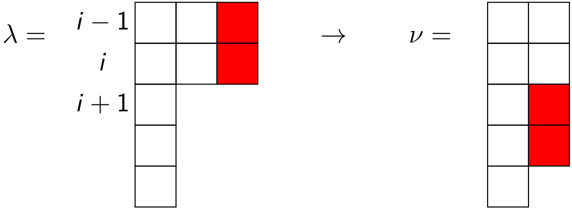


2. If $\lambda_i - \lambda_{i+1} > 1$, then we place that domino at the ends of rows $i+1$ and $i+2$.



Some more details

In order to prove that $M(\lambda) \geq m(\nu)$ We prove that either

$$0 \leq m(\nu) - m(\lambda) \leq 4$$

or

$$M(\nu) - M(\lambda) = 2$$

Example

How to conclude $M(\lambda) \geq m(\nu)$ from $0 \leq m(\nu) - m(\lambda) \leq 4$?

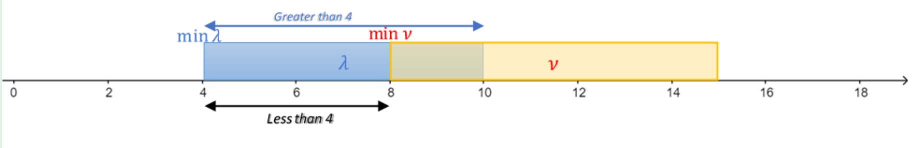
- In the picture below:

$$\text{Blue part} = m(\nu) - m(\lambda) \leq 4$$

$$\text{Blue} + \text{middle part} = M(\lambda) - m(\lambda) \geq 4$$

We conclude:

$$\text{Middle part} = M(\lambda) - m(\nu) \geq 0$$



Thank you for your attention!!