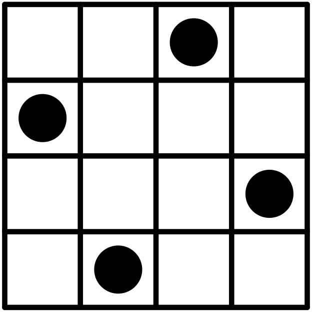
The Hertzsprungs problem

- In how many ways can one put n kings on an $n \times n$ chessboard such that each row and each column contains one and only one king and the kings are not attacking each other?

In the language of permutations

- S_n is the symmetric group on n elements.
- The plot of a permutation $\pi \in S_n$ is the set of all lattice points of the form (i, σ_i) where $1 \leq i \leq n$.



In the language of permutations

- Find the number of permutations $\sigma \in S_n$ such that for each $1 < i \leq n$, $|\sigma_i - \sigma_{i-1}| > 1$.

King permutations

- K_n is the set of all 'king' permutations in S_n.
- $K_1 = S_1$.
- $K_2 = K_3 = \emptyset$.
- $K_4 = \{[3142], [2413]\}$.
- K_n is closed to the reverse and inverse actions.

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Enumerative data

- An explicit formula for the number of king permutations was given by Robbins. He also showed that when n tends to infinity, the probability of picking such a permutation from S_n approaches e^{-2}.
- This set is counted in OEIS A002464.

The poset of king permutations

Definition

Let $\sigma, \pi \in \bigcup_{n\in\mathbb{N}} S_n$. We say that σ *contains* π if there is a sub-sequence of elements of σ that is order-isomorphic to π.

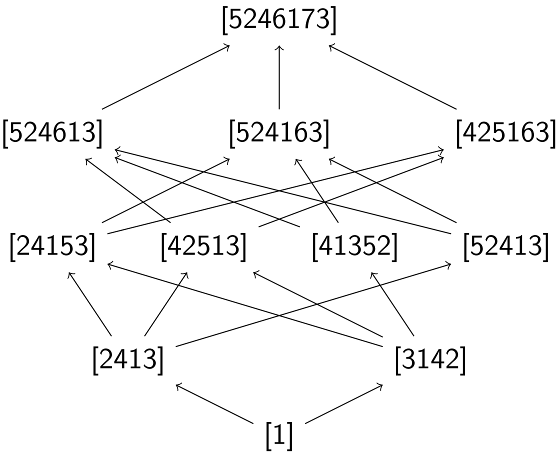
Example

[3624715] contains [3142] as the sub-sequences 6275 and 6475.

Definition

If σ is contained in π, then we write $\sigma \preceq \pi$. If $\sigma \preceq \pi$ but $\sigma \neq \pi$ then we write $\sigma \prec \pi$.

Example



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Regents

Definition

A king permutation $\sigma \in K_{n-1}$ is called a *regent* of $\pi \in K_n$ if $\sigma \prec \pi$.

Example

$[5246\mathbf{1}3]$

$\downarrow$

$[41\mathbf{3}52]$

$\downarrow$

$[3142]$

Some interesting questions about the regents

- Are there kings without regents?
- If there are, what can be said about their structure?
- Are there kings in K_n with n regents?
- If positive, what is their structure?

Separators

In order to investigate the structure of the poset of king permutations we introduce a new concept called **separator**

Definition

For $\sigma = [\sigma_1, \ldots, \sigma_n] \in S_n$ we say that $\sigma_{\mathbf{i}}$ *separates* σ_{j_1} from σ_{j_2} in σ if by omitting σ_i from σ we get a **new** $2-$block. This happens if and only if one of the following cases holds:

Separators Definition

Definition

1. Separator of type I :j_1, i, j_2 are subsequent numbers and $|\sigma_{j_1} - \sigma_{j_2}| = 1$, i.e

$$\sigma = [\ldots, \mathbf{a}, \mathbf{b}, \mathbf{a} \pm \mathbf{1}, \ldots]$$

2. Separator of type II: $\sigma_{j_1}, \sigma_i, \sigma_{j_2}$ are subsequent numbers and $|j_1 - j_2| = 1$, i.e,

$$\sigma = [\ldots, \mathbf{a}, \ldots, \mathbf{a} \pm \mathbf{1}, \mathbf{a} \mp \mathbf{1}, \ldots]$$

or

$$\sigma = [\ldots, \mathbf{a} \pm \mathbf{1}, \mathbf{a} \mp \mathbf{1}, \ldots, \mathbf{a}, \ldots].$$

Separators Example

Example

Let $\sigma = [132465879]$.
Then $Sep_I(\sigma) = \{3, 2, 6, 7\}$,
and $Sep_{II}(\sigma) = \{3, 2, 5, 8\}$. Thus $Sep(\sigma) = \{3, 2, 5, 6, 7, 8\}$ and $sep(\sigma) = |Sep(\sigma)| = 6$.
Note that 7 is a separator of type I, even though 7 is a part of a 2−block: 87, since by omitting 7 from σ we get a **new** 2−block: 78.

Some remarks about the separators

1. Notice the significance of the word 'new' in Definition 6. For example, the identity permutation has plenty of 2-blocks even though it has no separators.
2. The numbers 1 and n can only be separators of type I, σ_1 and σ_n can only be separators of type II.
3. If σ_i is a separator of type I in σ then i is a separator of type II in σ^{-1}. Hence $Sep_I(\sigma) = Sep_{II}(\sigma^{-1})$
4. $Sep_I(\sigma) = Sep_I(\sigma^r)$ and $Sep_{II}(\sigma) = Sep_{II}(\sigma^r)$ where σ^r is the reverse of σ.

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Kings Building Blocks

Theorem

For every $\pi \in K_n$ *(*$n \geq 4$*), either* $[2413] \preceq \pi$ *or* $[3142] \preceq \pi$*.*

Proof

- Use induction on n. Basis is trivial since $K_4 = \{[2413], [3142]\}$.
- Assume to the contrary that $\pi \in K_n$ contains neither [2413] nor [3142].
- Remove 1 from π to get a permutation which by induction hypothesis is not a king.
- 1 must be a separator of type I.
- w.l.o.g.

$$\pi = [\cdots, a, 1, a+1, \cdots].$$

- $a > 2$.
- Where is 2 located?
- If 2 is located right to the triple $a, 1, a+1$ then we are done since we have $\pi = [\cdots, a, 1, a+1, \cdots, 2, \cdots]$ which contains [3142].

Proof, Cntd.

- If 2 is located left to the triple then

$$\pi = [\cdots, 2, \cdots, a, 1, a+1, \cdots].$$

- Remove $a+1$ from π to get $\tau \notin K_{n-1}$.
- $a+1$ must be a separator of type *II* in π since 2 is far left.
- We have

$$\pi = [\cdots, \mathbf{2}, \cdots, \mathbf{a+2}, a, \mathbf{1}, \mathbf{a+1}, \cdots]$$

which contains [2413].

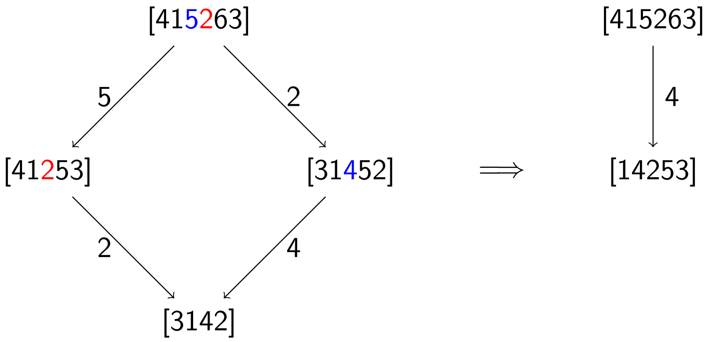
□

If you have a grandson you also have a regent

Theorem

Let $\sigma \in K_n$ with $n > 4$, and let $\pi \in K_{n-2}$ be such that $\pi \prec \sigma$. Then there exists $\tau \in K_{n-1}$ such that $\pi \prec \tau \prec \sigma$.

Example



block decomposiiton

Definition

Let $\pi = [\pi_1, \ldots, \pi_n] \in S_n$. A *block* (or *interval*) of π is a nonempty contiguous sequence of entries $\pi_i \pi_{i+1} \ldots \pi_{i+k}$ whose values also form a contiguous sequence of integers.

Example

If $\pi = [2647513]$ then 6475 is a block but 64751 is not.

Each permutation can be decomposed into singleton blocks, and also forms a single block by itself; these are the *trivial blocks* of the permutation. All other blocks are called *proper*.

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Inflation

- $\sigma = [67183524]$ can be decomposed as 67 1 8 3524.
- The relative order between the blocks forms the permutation [3142], i.e., if we take for each block one of its digits as a representative then the sequence of representatives is order-isomorphic to [3142].
- The block 67 is order-isomorphic to [12], and the block 3524 is order-isomorphic to [2413]. These are instances of the concept of *inflation*, defined as follows.

Definition

Let $n_1, \ldots, n_k$ be positive integers with $n_1 + \cdots + n_k = n$. The *inflation* of a permutation $\pi \in S_k$ by the permutations $\alpha_i \in S_{n_i}$ $(1 \leq i \leq k)$ is the permutation $\pi[\alpha_1, \ldots, \alpha_k] \in S_n$ obtained by replacing the i-th entry of π by a block which is order-isomorphic to the permutation α_i on the numbers $\{s_i + 1, \ldots, s_i + n_i\}$ instead of $\{1, \ldots, n_i\}$, where $s_i = n_1 + \cdots + n_{i-1}$ $(1 \leq i \leq k)$.

Example

The inflation of [2413] by [213], [21], [132] and [1] is

$$2413[213, 21, 132, 1] = [546\ 98\ 132\ 7].$$

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Kings without regents

Theorem

The following conditions are equivalent for each $\pi \in K_n$ with $n \geq 4$.

1. π *has no regents.*
2. *For each* $i \in \{1, \ldots, n\}$*, by removing* i *from* π*, we get a block of length* 3*.*
3. *There are* $\pi^1, \ldots, \pi^k \in \{[3142], [2413]\}$ *and* $\pi' \in S_k$ *such that* $\pi = \pi'[\pi^1, \ldots, \pi^k]$*.*

Example

$$\pi = [7, 5, 8, 6, 2, 4, 1, 3, 10, 12, 9, 11] \in K_{12}.$$

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$$\pi = [7, 5, 8, 6, 2, 4, 1, 3, 10, 12, 9, 11] \in K_{12}.$$

$$= 213[3142, 2413, 2413]$$

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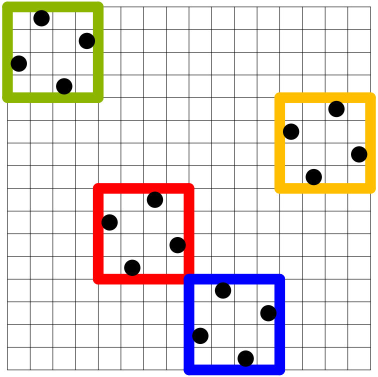

Figure: The plot of [14, 16, 13, 15, 7, 5, 8, 6, 2, 4, 1, 3, 11, 9, 12, 10]

Regents enumeration

Corollary

The number of permutations in K_n which have no regents is:

$$\begin{cases} 2^k k! & n = 4k \\ 0 & O.W. \end{cases}$$

The descendants of king permutations without regents

Corollary

Let σ be as above. Then for each $\pi \in K_l$ $(l < n)$ such that $\pi \prec \sigma$ we have $\pi = \pi'[\pi^1, \ldots, \pi^m]$, $(m \leq k)$, where $\pi^i \in \{[3142], [2143], [1]\}$ and $\pi' \in S_m$ is such that $\pi' \prec \sigma'$.

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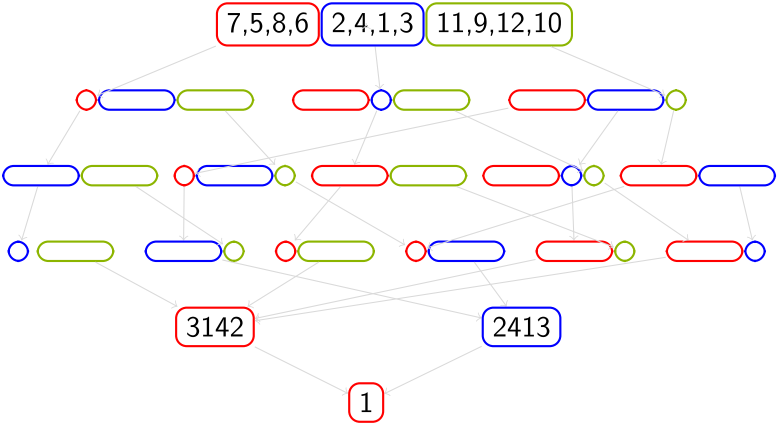
Example: The poset of 213[3142,2413,3142], king without regent



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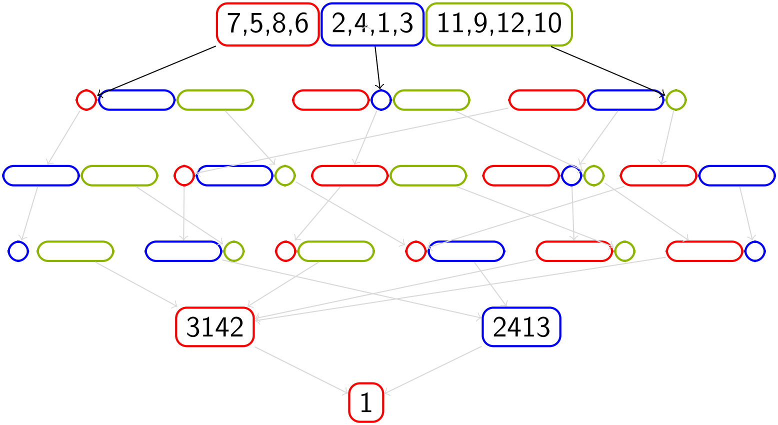

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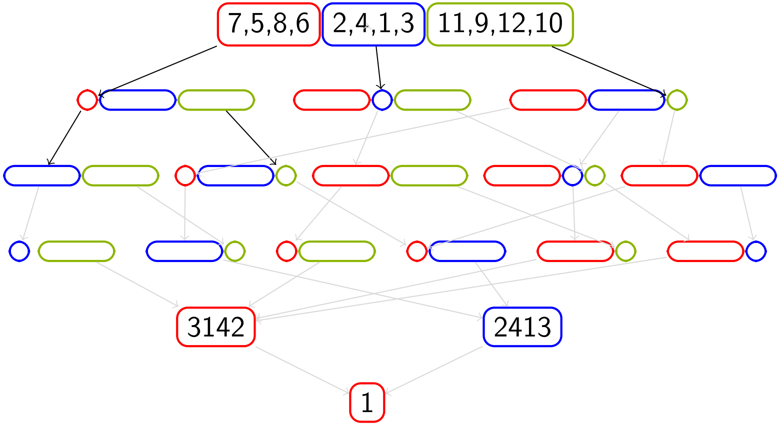

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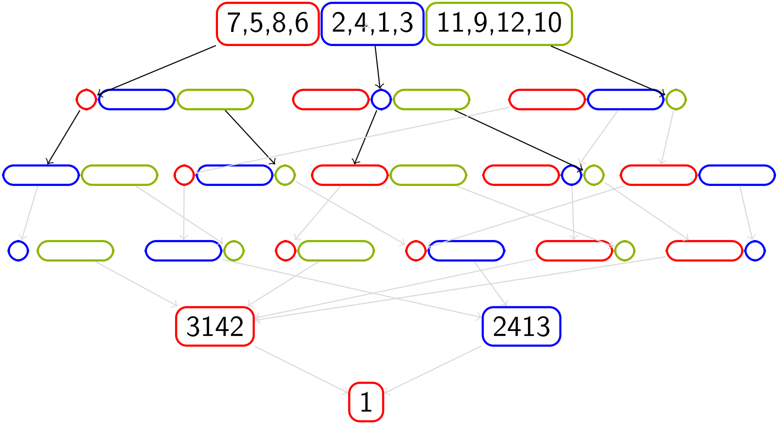

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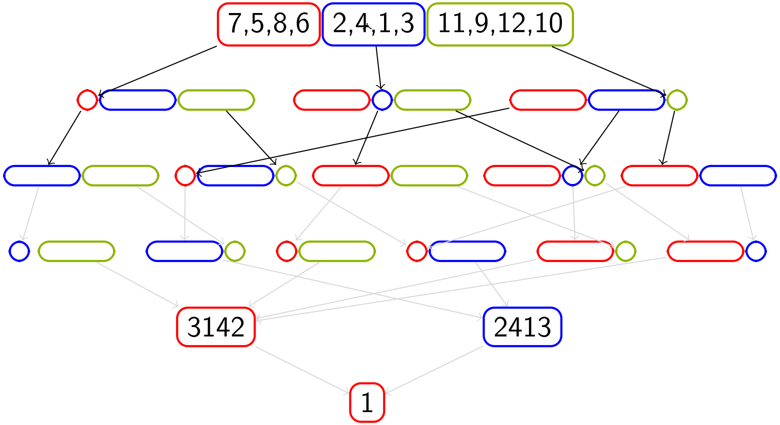

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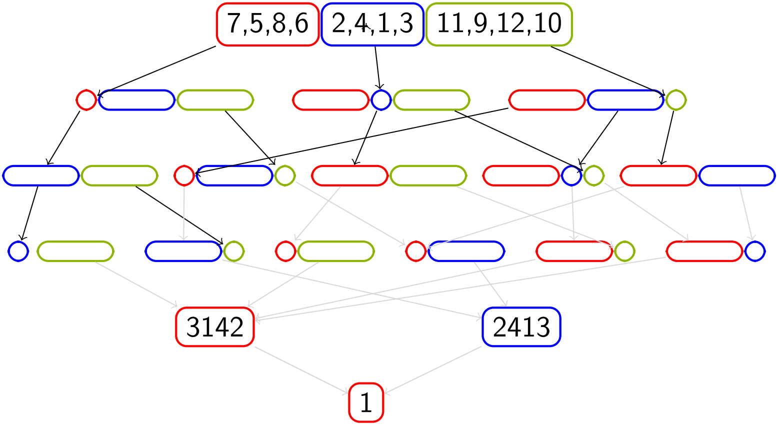

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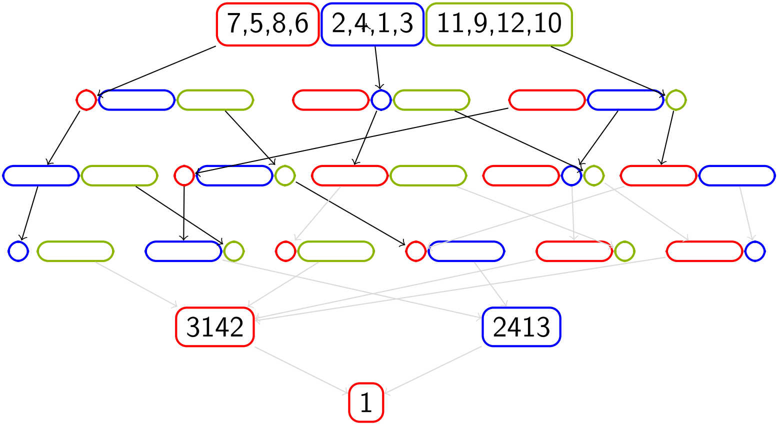

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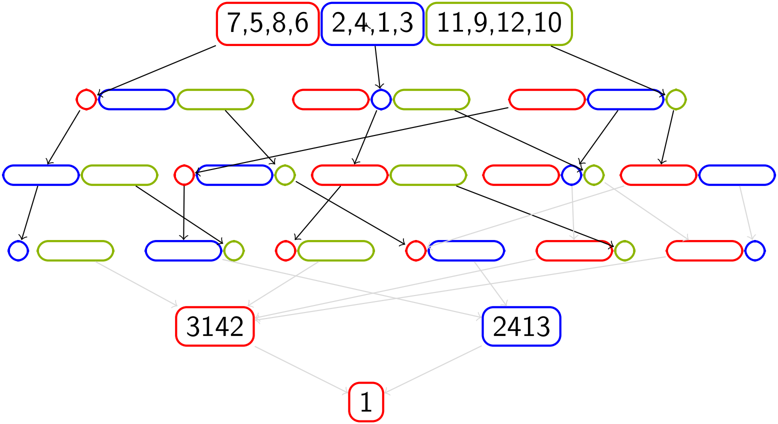

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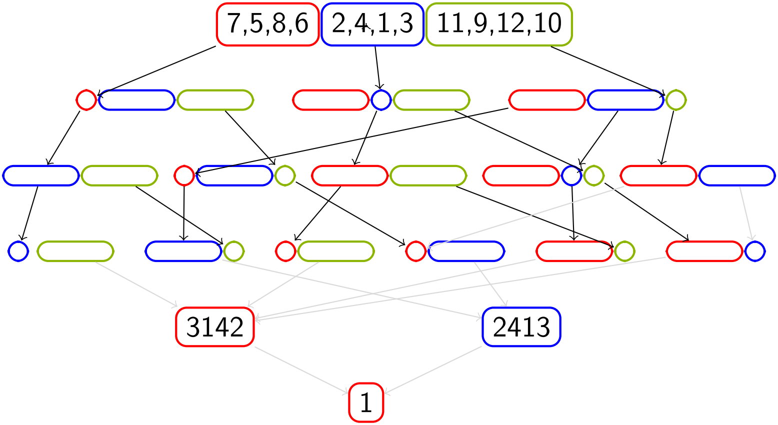

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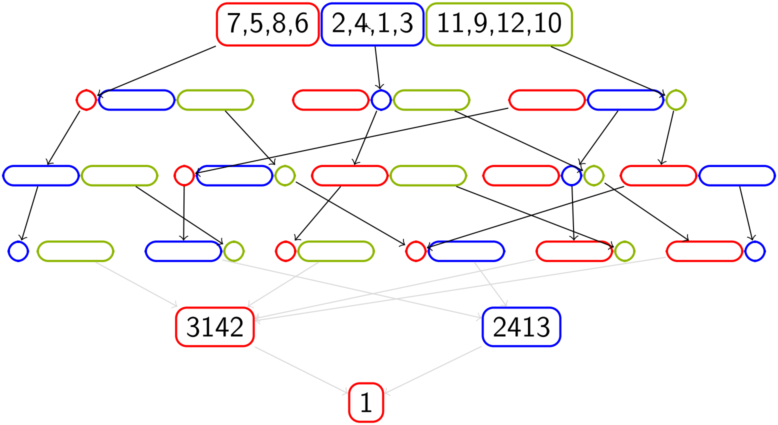

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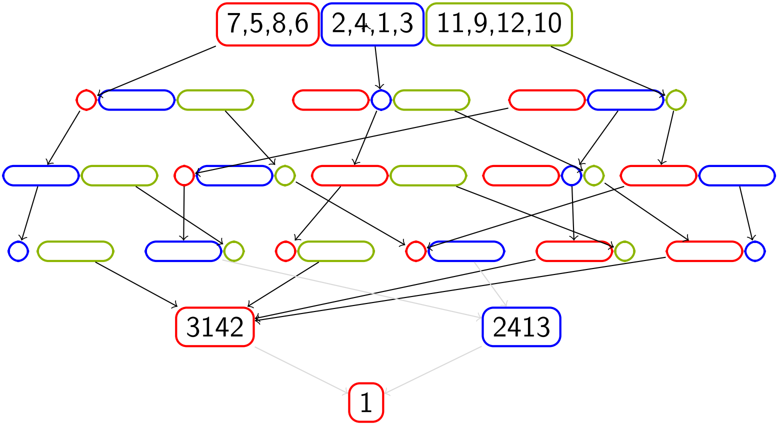

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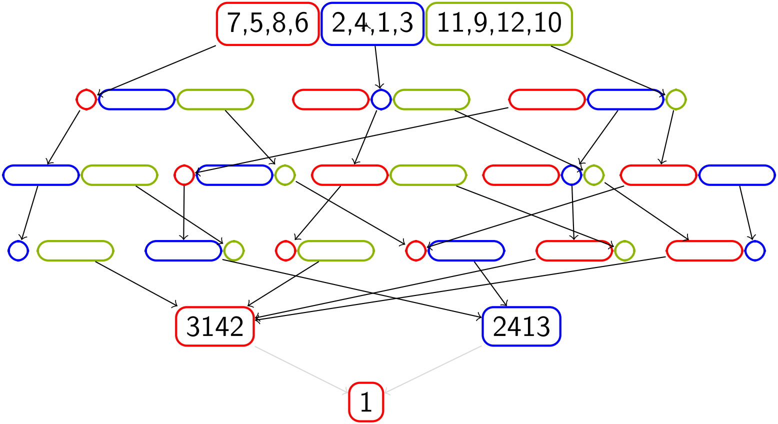

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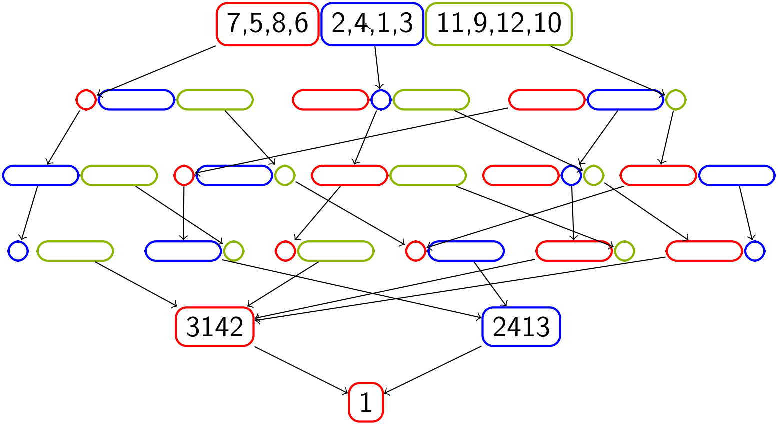


Another property of our poset

Theorem

Let $n > 4$. For each $\sigma \in K_n$ there exists $\pi \in K_5$ s.t. $\pi \preceq \sigma$.

The Möbious function of the Kings poset

Definition

The closed interval $[\tau, \sigma]$ is defined as:

$$[\tau, \sigma] = \{\pi \in \mathcal{K} \mid \tau \preceq \pi \preceq \sigma\}.$$

The half open interval is defined as:

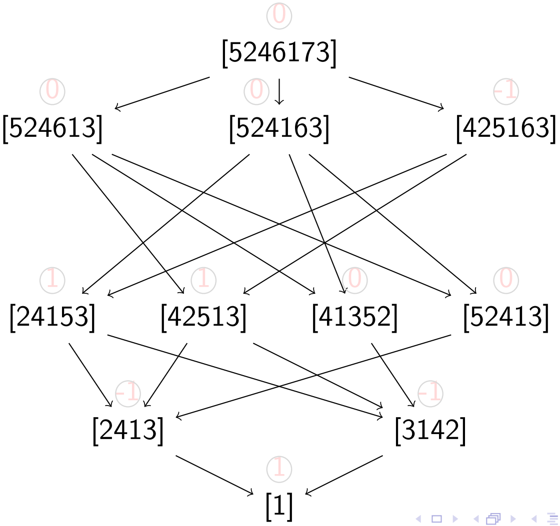
$$[\tau, \sigma) = \{\pi \in \mathcal{K} \mid \tau \preceq \pi \prec \sigma\}.$$

Definition

$$\mu(\tau,\sigma) = \begin{cases} 0, & \text{if } \tau \not\leq \sigma; \\ 1, & \text{if } \tau = \sigma; \\ -\sum_{\pi \in [\tau,\sigma)} \mu(\tau,\pi), & \text{Otherwise.} \end{cases}$$

If $\tau = [1]$, the identity permutation of length 1, then we write $\mu(\pi) := \mu([1],\pi)$.

Example



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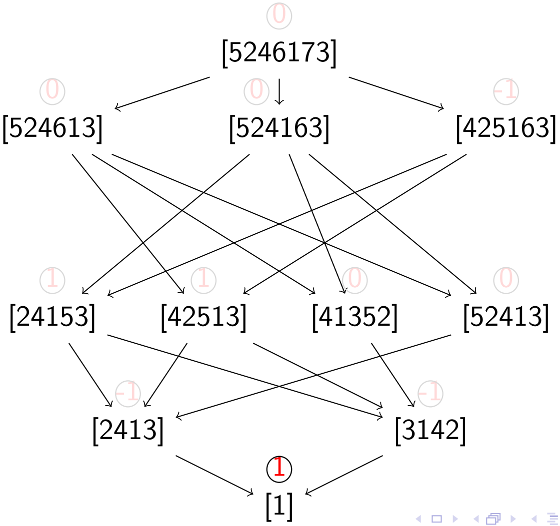

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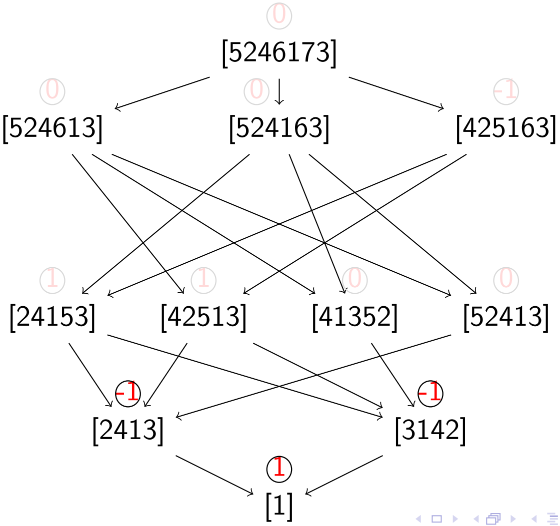
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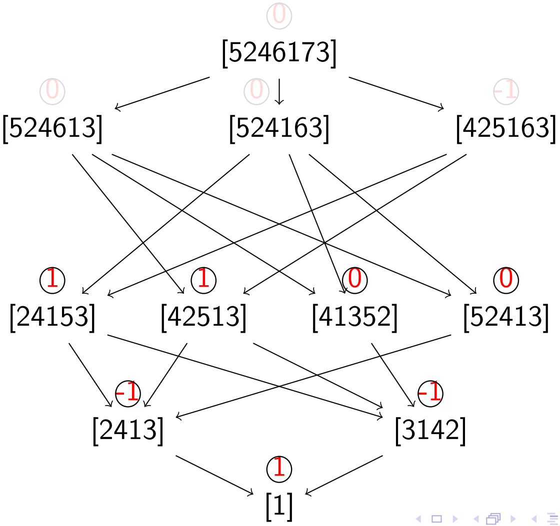
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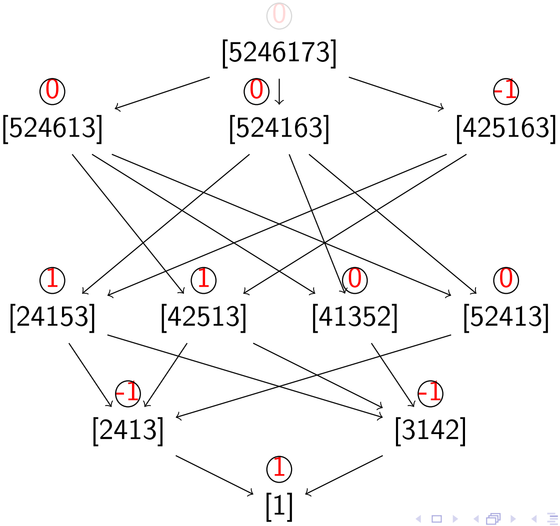
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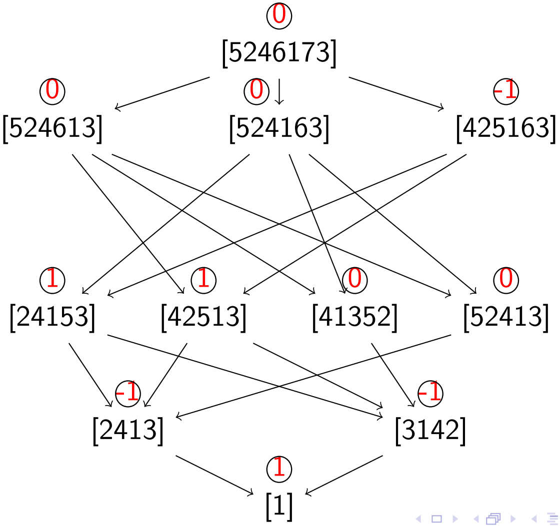
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Theorem

Let $\pi \in K_n$*, with* $n > 4$*. If* $[2413] \not\leq \pi$ *or* $[3142] \not\leq \pi$ *then* $\mu(\pi) = 0$ *in* K_n*.*

Let

$$\mathbb{G} = \{[24153], [35142], [42513], [31524]\}.$$

It is easy to see that $\mathbb{G}$ consists of all the elements of K_5 which contain both [2413] and [3142].

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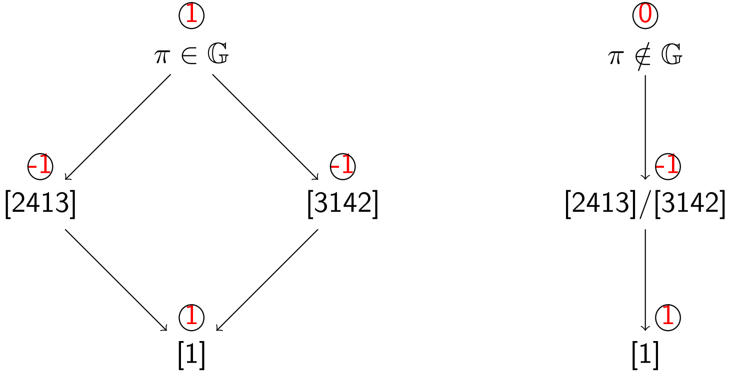
Note that in K_5, $\mu(\pi) = 1$ if and only if $\pi \in \mathbb{G}$ (otherwise $\mu(\pi) = 0$).



Theorem

Let $\pi \in K_n$ *with* $n > 5$ *such that there is* **only one** $\sigma \prec \pi$ *such that* $\sigma \in \mathbb{G}$ *and for each* $\sigma' \prec \pi$ *such that* $\sigma \not\prec \sigma'$ *we have either* σ' *avoids* [3142], *or* σ' *avoids* [2413]. *Then in the poset of king permutations* $\mu(\pi) = 0$.

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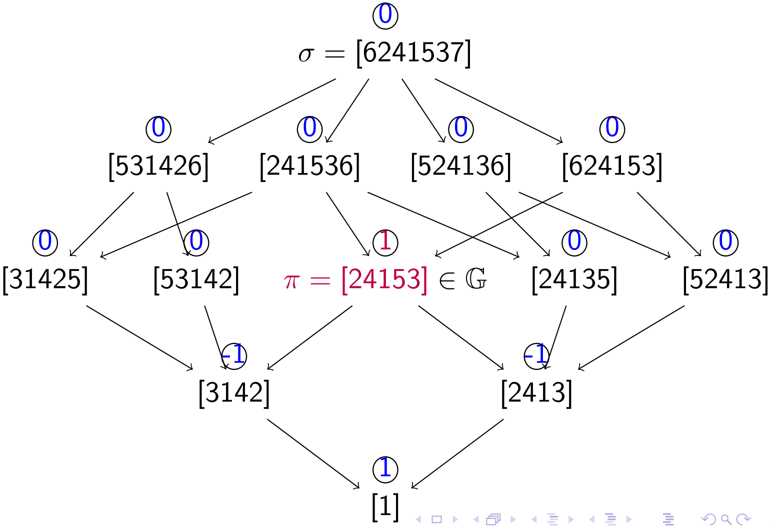
Example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